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Kummer Surface with D_4-Symmetry

Isao Naruki

For the simple root system D_4 there are exactly three linearly independent Weyl-group-invariant homogeneous polynomials of degree 4 on the Cartan subalgebra V. Since V is 4-dimensional, the null locus S of such an polynomial $\neq 0$ is a quartic surface in the associated projective space $\mathbf{P}(V) \cong \mathbf{P}_3(\mathbf{C})$. ($S$ has two parameters.) S is smooth in general. In this note however we will only discuss a special case where S is a Kummer quartic i.e. quartic surface with 16 nodes (ordinary double points). This case is introduced by imposing the following condition on S:

(A) *Some (hence any by invariance) root-section of S decomposes into two conics intersecting transversally.*

For any root r the *section* of S by r is the intersection of S and the null plane $H_r := \{(x) \in \mathbf{P}(V) : r(x) = 0\}$. (This plane curve is in general irreducible.) From now on we assume that S satisfies (A), so S is now a Kummer surface.

S has still one parameter. Explicitly S is given by the equation

$$I_1(x) - (s^2+1)I_2(x) + 2s(s^2+3)I_3(x) = 0$$

where s ($s^2+3 \neq 0, s = \pm 1$) is the parameter, $I_1(x) := \sum_{i=1}^4 x_i^4$, $I_2(x) := \sum_{1 \leq i<j \leq 4} x_i^2 x_j^2$, $I_3(x) := x_1x_2x_3x_4$ and the coordinates (x_1, x_2, x_3, x_4) are so chosen that the roots are $\pm(x_i \pm x_j)$. The Weyl group is generated by the even sign changes and permutations of x_1, x_2, x_3, x_4. The 16 nodes are the orbit of $(s, 1, 1, 1)$. We see that the 16 nodes lie four by four on the 12 root-sections to be the inter-section points of the conics in (A). Each node is on exactly three root-sections.

For the definiteness of argument we fix a root r and let C_1, C_2 be the conics such that $C_1 \cup C_2 = H_r \cap S$. Let $\{q_0, q_1, q_2, q_3\} = C_1 \cup C_2$. Recall now that the abelian surface

$\mathcal{A}$ associated with S is the double cover of S branched over the 16 nodes; so the nodes are naturally imbedded into $\mathcal{A}$; in particular $\{q_0, q_1, q_2, q_3\} \subseteq \mathcal{A}$. We regard q_0 as the zero of $\mathcal{A}$. We remark that the inverse images E_1, E_2 of C_1, C_2 by $\mathcal{A} \to S$ are elliptic curves. They are thus two subgroups of $\mathcal{A}$ such that $E_1 \cup E_2 = \{q_0, q_1, q_2, q_3\}$. We set $G_0 := E_1 \cap E_2$. This is a subgroup of the 2-torsion $\mathcal{A}(2)$ of $\mathcal{A}$. We also form the diagonal group $\Delta_0 := \{(q_i, q_i)\}_{i=0,1,2,3}$ in the product group $\mathcal{E} := E_1 \times E_2$.

Proposition 1. *The product mapping $\mathcal{E} = E_1 \times E_2 \ni (x, y) \mapsto xy \in \mathcal{A}$ induces the isomorphism*

$$\mathcal{E}/\Delta_0 \cong \mathcal{A}. \tag{1}$$

It follows also

$$\mathcal{A}/G_0 \cong \mathcal{E}. \tag{2}$$

Remark. So far we have only used the existence of a plane which cuts from a quartic two conics in a transversal position. This property is therefore a characterization of elliptic Kummer surfaces of degree 2.

We call such an isomorphism as (1) an *almost product structure* on $\mathcal{A}$; (1) depends on the root r fixed above. Since there are 12 roots of D_4 up to sign, we have 12 almost product structures for $\mathcal{A}$. But not all of them are different.

Proposition 2. *The almost product structures associated with two roots are identical if and only if they are orthogonal (with respect to the Killing form $\sum_{i=1}^{4} x_i^2$).*

The existence of different almost product structures suggests that the original D_4-symmetry should be explained by the symmetry of $\mathcal{A}$ i.e. its non-trivial endomorphisms. This leads further to the natural question: what is the relation between the moduli of two elliptic curves E_1 and E_2 which should exist since we have only one parameter s. The stabilizer of the Weyl symmetry at q_0 is isomorphic to S_3, so it contains an element of order 3. This fact proves

Proposition 3. *There is an isogeny of degree 3 between E_1 and E_2.*

By this result we can describe E_1 and E_2 by two lattices L_1, L_2 in $\mathbf{C}$ in the following way:

$$3L_2 \subset L_1 \subset L_2, \quad [L_2 : L_1] = 3. \tag{3}$$

$$E_1 = \mathbf{C}/L_1, \quad E_2 = \mathbf{C}/L_2. \tag{4}$$

Then, by (1), we have also the isomorphism

$$(\mathbf{C} \times \mathbf{C})/L \cong \mathcal{A} \tag{5}$$

where L is a lattice in $\mathbf{C} \times \mathbf{C}$ such that $2L \subset L_1 \times L_2 \subset L$, $[L : L_1 \times L_2] = 4$.

Proposition 4. *The lattice in (5) is given by*

$$L = \{(a,b) \in \mathbf{C} \times \mathbf{C} : 2a \in L_1, 2b \in L_2, a - b \in L_2\}.$$

The stabilizer at q_0 is lifted to a subgroup of Aut($\mathcal{A}$) *generated by the elements which are induced by the matrices*

$$M := \begin{pmatrix} \frac{1}{2} & \frac{3}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \quad N := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Check that $ML = L$, $NL = L$ and that $M^3 = -1$, $N^2 = (MN)^2 = 1$. We close this note by remarking that the entire D_4-symmetry is generated by the stabilizer described above and the (translation) action of $\mathcal{A}(2)$ over $S = \mathcal{A}/\{\pm 1\}$.

The analytic counterpart of this story contains the parametric representation of S by the Weierstrass σ-functions associated with E_1 and E_2; it also contains the explanation of the parameter s and the isogeny between the elliptic curves by some modular models. This interesting topic will however be published elsewhere in a more general form.